Inductive Construction of Nilpotent Modules of Quantum Groups at Roots of Unity

Yuuki Abe *
Department of Mathematics,
Sophia University

Abstract

The purpose of this paper is to prove that we can construct all finite dimensional irreducible nilpotent modules of type 1 inductively by using Schnizer homomorphisms for quantum algebra at roots of unity of type A_n, B_n, C_n, D_n or G_2.

1 Introduction

Let $U_q(\mathfrak{g})$ be the quantum algebra of a finite dimensional complex simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. The theory of $U_q(\mathfrak{g})$-modules is almost same as the one of $\mathfrak{g}$ if q is not a root of unity. But, if q is a root of unity, it is quite different from the one of $\mathfrak{g}$. For example, there are the following differences.

- Finite dimensional modules are not always semisimple.
- Finite dimensional irreducible modules are not necessarily highest or lowest weight modules.
- Among finite dimensional irreducible modules, there exist maximum dimensional modules.

The theory of $U_q(\mathfrak{g})$-modules at roots of unity is introduced in [4].

Let ε be a primitive l-th root of unity. The completely classification of finite dimensional irreducible $U_\varepsilon(\mathfrak{g})$-modules is not given yet. But, the classification of finite dimensional irreducible *nilpotent* $U_\varepsilon(\mathfrak{g})$-modules of type 1 is already given by Lusztig in [6], [7] (see §3). In particular, it is known that these modules are classified by highest weights.

In [8], Nakashima discover that we can construct these modules by using the modules introduced in [3] if $\mathfrak{g}$ is type A_n. Moreover, in [1], we discover that we can construct these modules by using the *Schnizer modules* introduced in [9] if $\mathfrak{g}$ is type B_n, C_n or D_n.

In this paper, we construct these modules inductively in the case that $\mathfrak{g}$ is type A_n, B_n, C_n, D_n or G_2 by using the Schnizer homomorphisms introduced in [10]. Then we can construct finite dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g})$-modules of type 1 with highest weight $(0,\cdots,0,\lambda_k,\cdots,\lambda_n)$ as a submodule of a $l^{(N_n-N_{k-1})}$-dimensional $U_\varepsilon(\mathfrak{g})$-module, where N_n is the number of the positive roots of $\mathfrak{g}$ and n is the rank of $\mathfrak{g}$. In particular, these results cover the ones of [1].

The organization of this paper is as follows. In §2, we review the quantum algebras at roots of unity. In §3, we introduce the nilpotent modules and their classification theorem. In §4, we introduce the Schnizer homomorphisms. Finally, in §5-§7, we give inductive construction of all finite dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g})$-modules of type 1 in the case of $\mathfrak{g} = A_n$, B_n, C_n, D_n or G_2.

*e-mail: yu-abe@sophia.ac.jp

2 Quantum algebras at roots of unity

We fix the following notations. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ of type A_n, B_n, C_n, D_n or G_2. We set $I := \{1, 2, \cdots, n\}$. Let $\{\alpha_i\}_{i\in I}$ be the set of simple roots, Δ be the set of roots and Δ_+ be the set of positive roots of $\mathfrak{g}$. Let N be the number of positive roots of $\mathfrak{g}$, that is, $N = \frac{1}{2}n(n+1)$ (resp. n^2, n^2, $(n-1)n$, 6) if $\mathfrak{g} = A_n$ (resp. B_n, C_n, D_n, G_2). We define the root lattice $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and the positive loot lattice $Q_+ := \bigoplus_{i\in I} \mathbb{Z}_+\alpha_i$, where $\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$. Let $(\mathtt{a}_{i,j})_{i,j\in I}$ be the Cartan matrix associated with $\mathfrak{g}$ such that

$$\begin{cases} \mathtt{a}_{1,2} = -2 & \mathfrak{g} = B_n, \\ \mathtt{a}_{2,1} = -2 & \mathfrak{g} = C_n, \\ \mathtt{a}_{1,2} = 0,\ \mathtt{a}_{1,3} = \mathtt{a}_{2,3} = -1 & \mathfrak{g} = D_n, \\ \mathtt{a}_{1,2} = -3 & \mathfrak{g} = G_2. \end{cases}$$

We define $(d_1, \cdots, d_n) := (1, \cdots, 1)$ (resp. $(\frac{1}{2}, 1, \cdots, 1)$, $(2, 1, \cdots, 1)$, $(1, \cdots, 1)$, $(1, 3)$) if $\mathfrak{g} = A_n$ (resp. B_n, C_n, D_n, G_2). We denote the Weyl group of $\mathfrak{g}$ by $\mathcal{W}$ which is generated by the simple reflections $\{s_i\}_{i\in I}$.

Let l be an odd integer which is greater than 2. We assume that l is not divisible by 3 if $\mathfrak{g} = G_2$. Let ε (resp. $\varepsilon^{\frac{1}{2}}$) be a primitive l-th root of unity if $\mathfrak{g} \neq B_n$ (resp. $\mathfrak{g} = B_n$). For $r \in \mathbb{Z}$, $m \in \mathbb{N}$, $d \in \mathbb{Q}$ such that $\varepsilon^{2d} \neq 1$, we define

$$[r]_{\varepsilon^d} := \frac{\varepsilon^{dr} - \varepsilon^{-dr}}{\varepsilon^d - \varepsilon^{-d}}, \quad [r] := [r]_\varepsilon,$$
$$[m]_{\varepsilon^d}! := [m]_{\varepsilon^d}[m-1]_{\varepsilon^d} \cdots [1]_{\varepsilon^d}, \quad [0]_{\varepsilon^d}! := 1.$$

Definition 2.1. *The quantum algebra $U_\varepsilon(\mathfrak{g})$ is an associative $\mathbb{C}$-algebra generated by $\{e_i, f_i, t_i^{\pm 1}\}_{i\in I}$ with the relations*

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i t_j = t_j t_i,$$
$$t_i e_j t_i^{-1} = \varepsilon_i^{\mathtt{a}_{i,j}} e_j, \quad t_i f_j t_i^{-1} = \varepsilon_i^{-\mathtt{a}_{i,j}} f_j,$$
$$e_i f_j - f_j e_i = \delta_{i,j}\{t_i\}_{\varepsilon_i},$$
$$\sum_{k=0}^{1-\mathtt{a}_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-\mathtt{a}_{i,j}-k)} = \sum_{k=0}^{1-\mathtt{a}_{i,j}} (-1)^k f_i^{(k)} f_j f_i^{(1-\mathtt{a}_{i,j}-k)} = 0 \quad i \neq j,$$

where

$$e_i^{(k)} := \frac{1}{[k]_{\varepsilon_i}!} e_i^k, \quad f_i^{(k)} := \frac{1}{[k]_{\varepsilon_i}!} f_i^k, \quad \{t_i\}_{\varepsilon_i} := \frac{t_i - t_i^{-1}}{\varepsilon_i - \varepsilon_i^{-1}}, \quad \varepsilon_i := \varepsilon^{d_i}.$$

Let $U_\varepsilon^+(\mathfrak{g})$ (resp. $U_\varepsilon^-(\mathfrak{g})$, $U_\varepsilon^0(\mathfrak{g})$) be the $\mathbb{C}$-subalgebra of $U_\varepsilon(\mathfrak{g})$ generated by $\{e_i\}_{i\in I}$ (resp. $\{f_i\}_{i\in I}$,$\{t_i^{\pm 1}\}_{i\in I}$). Moreover, we extend the algebra by adding the elements $\{t_i^{\pm\frac{1}{k}} \,|\, i, k \in I\}$.

Let w_0 be a longest element of $\mathcal{W}$ and $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced expression of w_0. We set

$$\beta_1 := \alpha_{i_1},\ \beta_2 := s_{i_1}(\alpha_{i_2}),\ \cdots,\ \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}).$$

Indeed, we have $\Delta_+ = \{\beta_1, \cdots, \beta_N\}$. Then there exist the vectors $\{e_{\beta_i}\}_{i=1}^N$, $\{f_{\beta_i}\}_{i=1}^N$ in $U_\varepsilon(\mathfrak{g})$ which are called "root vectors" (cf. [4], [7]), where $e_{\alpha_i} = e_i$, $f_{\alpha_i} = f_i$ for $i \in I$. These vectors satisfy the following properties.

Proposition 2.2 ([4] Proposition 1.7, [7]). *(i) $\{e_{\beta_1}^{m_1} \cdots e_{\beta_N}^{m_N} | m_1, \cdots, m_N \in \mathbb{Z}_+\}$ is a $\mathbb{C}$-basis of $U_\varepsilon^+(\mathfrak{g})$.*

(ii) $\{f_{\beta_1}^{m_1}\cdots f_{\beta_N}^{m_N}|m_1,\cdots,m_N\in\mathbb{Z}_+\}$ *is a* $\mathbb{C}$*-basis of* $U_\varepsilon^-(\mathfrak{g})$.

(iii) $\{t_1^{m_1}\cdots t_n^{m_n}|m_1,\cdots,m_n\in\mathbb{Z}\}$ *is a* $\mathbb{C}$*-basis of* $U_\varepsilon^0(\mathfrak{g})$.

(iv) *Let* $\phi: U_\varepsilon^-(\mathfrak{g})\otimes U_\varepsilon^0(\mathfrak{g})\otimes U_\varepsilon^+(\mathfrak{g})\longrightarrow U_\varepsilon(\mathfrak{g})$ $(u_-\otimes u_0\otimes u_+\mapsto u_-u_0u_+)$ *be the multiplication map. Then* ϕ *is an isomorphism of* $\mathbb{C}$*-vector space.*

Let $Z(U_\varepsilon(\mathfrak{g}))$ be the center of $U_\varepsilon(\mathfrak{g})$.

Proposition 2.3 ([4] Corollary 3.1). *We have* $\{e_\alpha^l, f_\alpha^l, t_i^l\,|\,\alpha\in\Delta_+, i\in I\}\subset Z(U_\varepsilon(\mathfrak{g}))$.

Now, for $i\in I$, we set

$$\deg(e_i):=\alpha_i,\quad \deg(f_i):=-\alpha_i,\quad \deg(t_i):=0. \tag{2.1}$$

Obviously, these are compatible with the relations of $U_\varepsilon(\mathfrak{g})$. Therefore, we can regard $U_\varepsilon(\mathfrak{g})$ as Q-graded algebra, and we have

$$U_\varepsilon(\mathfrak{g})=\bigoplus_{\alpha\in Q}U_\varepsilon(\mathfrak{g})_\alpha,\quad U_\varepsilon(\mathfrak{g})_\alpha U_\varepsilon(\mathfrak{g})_{\alpha'}\subset U_\varepsilon(\mathfrak{g})_{(\alpha+\alpha')},$$

for $\alpha,\alpha^{'}\in Q$, where $U_\varepsilon(\mathfrak{g})_\alpha:=\{u\in U_\varepsilon|\deg(u)=\alpha\}$.

Proposition 2.4 ([5] §8). *We have* $e_\alpha\in U_\varepsilon^+(\mathfrak{g})\cap U_\varepsilon(\mathfrak{g})_\alpha$ *and* $f_\alpha\in U_\varepsilon^-(\mathfrak{g})\cap U_\varepsilon(\mathfrak{g})_{-\alpha}$ *for all* $\alpha\in\Delta_+$.

3 Nilpotent modules

Definition 3.1. *Let* L *be a* $U_\varepsilon(\mathfrak{g})$*-module. If* $e_\alpha^l=f_\alpha^l=0$ *on* L *for all* $\alpha\in\Delta_+$*, then we call* L *"nilpotent module". In particular, if* $t_i^l=1$ *on* L *for all* $i\in I$*, then we call* L *"nilpotent module of type 1".*

Remark 3.2. *Nilpotent* $U_\varepsilon(\mathfrak{g})$*-modules of type 1 are same as* $U_\varepsilon^{fin}(\mathfrak{g})$*-modules of type 1, where* $U_\varepsilon^{fin}(\mathfrak{g})$ *is the finite dimensional quantum algebra introduced in [6], [7] (see [1]).*

In general, finite dimensional irreducible $U_\varepsilon^{fin}(\mathfrak{g})$*-modules are divided into* 2^n *types according to* $\{\sigma: Q\longrightarrow\{\pm1\}$*; homomorphism of group* $\}$*. Without a loss of generality, we may assume that finite dimensional irreducible* $U_\varepsilon^{fin}(\mathfrak{g})$*-modules are of type* 1.

Definition 3.3. *Let* L *be a* $U_\varepsilon(\mathfrak{g})$*-module.*

(i) *We set* $P(L):=\{v\in L\,|\,e_iv=0$ *for all* $i\in I\}$ *and call the vectors in* $P(L)$ *"primitive vector".*

(ii) *Let* $\lambda=(\lambda_i)_{i\in I}\in\mathbb{C}^n$*. We assume that* L *is generated by a nonzero vector* $v_0\in P(L)$ *such that* $t_iv_0=\varepsilon_i^{\lambda_i}v_0$ *for all* $i\in I$*. Then we call* L *"highest weight module with highest weight* λ*" and* v_0 *"highest weight vector".*

Now, we introduce the classification theorem of finite dimensional irreducible nilpotent $U_\varepsilon(\mathfrak{g})$-modules of type 1. We set $\mathbb{Z}_l:=\{\lambda\in\mathbb{Z}\,|\,0\le\lambda\le l-1\}$.

Theorem 3.4 ([6], [7]). *For any* $\lambda\in\mathbb{Z}_l^n$*, there exists a unique (up to isomorphic) finite dimensional irreducible nilpotent* $U_\varepsilon(\mathfrak{g})$*-module* $L_\varepsilon^{nil}(\lambda)$ *of type 1 with highest weight* λ*. Conversely, if* L *is a finite dimensional irreducible nilpotent* $U_\varepsilon(\mathfrak{g})$*-module of type 1, then there exists a* $\lambda\in\mathbb{Z}_l^n$ *such that* L *is isomorphic to* $L_\varepsilon^{nil}(\lambda)$.

By the similar manner to the proof of Theorem 5.5(ii) in [8] or Theorem 4.10 in [1], we obtain the following proposition.

Proposition 3.5. *For* $\lambda\in\mathbb{Z}_l^n$*, let* L *be a nilpotent highest weight* $U_\varepsilon(\mathfrak{g})$*-module of type 1 with highest weight* λ*. We assume* $dim(P(L))=1$*. Then* L *is irreducible* $U_\varepsilon(\mathfrak{g})$*-module. In particular,* L *is isomorphic to* $L_\varepsilon^{nil}(\lambda)$ *as* $U_\varepsilon(\mathfrak{g})$*-module.*

4 Schnizer homomorphisms

In the rest of the paper, we denote $\mathfrak{g}$ by $\mathfrak{g}_n$ if the rank of $\mathfrak{g}$ is n and e_i, f_i, t_i in $U_\varepsilon(\mathfrak{g}_n)$ by $e_{i,n}, f_{i,n}, t_{i,n}$.

We fix the following notations. Let V_n be a l^n-dimensional $\mathbb{C}$-vector spase and $\{v_n(m_n)\,|\,m_n = (m_{1,n}, \cdots, m_{n,n}) \in \mathbb{Z}_l^n\}$ be a basis of V_n, where $\mathbb{Z}_l := \{m \in \mathbb{Z}\,|\,0 \le m \le l-1\}$. We set $v_n(m_n + lm_n^{'}) := v_n(m_n)$ for $m_n, m_n^{'} \in \mathbb{Z}_l^n$. For $i \in I$, we set

$$\epsilon_{i,n} := (\delta_{i,1}, \delta_{i,2}, \cdots, \delta_{i,n}) \in \mathbb{Z}_l^n, \tag{4.1}$$

where $\delta_{i,j}$ is the Kronecker's delta. For $i \in I$, $a_{i,n} \in \mathbb{C}^\times$, $b_{i,n} \in \mathbb{C}$, we define linear maps $x_{i,n}, z_{i,n} \in \mathrm{End}(V_n)$ by

$$x_{i,n} v_n(m_n) := a_{i,n} v_n(m_n - \epsilon_{i,n}), \quad z_{i,n} v_n(m_n) := \varepsilon^{m_{i,n}+b_{i,n}} v_n(m_n) \quad (m_n \in \mathbb{Z}_l^n). \tag{4.2}$$

For any $z \in \mathrm{End}(V_n)$ such that $z^{-1} \in \mathrm{End}(V_n)$ and $d \in \mathbb{Q}$ such that $\varepsilon^{2d} \neq 1$, we set

$$\{z\}_{\varepsilon^d} := \frac{z - z^{-1}}{\varepsilon^d - \varepsilon^{-d}}. \tag{4.3}$$

Then we have

$$\{z_{i,n}\}_{\varepsilon^d} v_n(m_n) = [d^{-1}(m_{i,n} + b_{i,n})]_{\varepsilon^d} v_n(m_n). \tag{4.4}$$

For any $\mathbb{C}$-vector space V, we regard $\mathrm{End}(V) \otimes U_\varepsilon(\mathfrak{g}_n)$ as $\mathbb{C}$-algebra by

$$(x \otimes u)(x^{'} \otimes u^{'}) := (xx^{'}) \otimes (uu^{'}) \quad (x, x^{'} \in \mathrm{End}(V), u, u^{'} \in U_\varepsilon(\mathfrak{g}_n)).$$

Theorem 4.1 ([10] Theorem 3.2, 4.10). *(a) Let $\lambda_n \in \mathbb{C}$, $a_n = (a_{i,n})_{i=1}^n \in (\mathbb{C}^\times)^n$, and $b_n = (b_{i,n})_{i=1}^n \in \mathbb{C}^n$. Then we obtain a $\mathbb{C}$-algebra homomorphism $\rho_n^A := \rho_n^A(a_n, b_n, \lambda_n) : U_\varepsilon(A_n) \longrightarrow End(V_n) \otimes U_\varepsilon(A_{n-1})$ such that*

$$\rho_n^A(e_{i,n}) = \{z_{i-1,n} z_{i,n}^{-1}\} x_{i,n} + x_{i-1,n}^{-1} x_{i,n} e_{i-1,n-1}, \tag{4.5}$$

$$\rho_n^A(t_{i,n}) = z_{i-1,n} z_{i,n}^{-2} z_{i+1,n} t_{i,n-1}, \tag{4.6}$$

$$\rho_n^A(f_{i,n}) = \{z_{i,n} z_{i+1,n}^{-1} t_{i,n-1}^{-1}\} x_{i,n}^{-1} + f_{i,n-1}, \tag{4.7}$$

where

$$t_{n,n-1} := \varepsilon^{-\lambda_n} \prod_{i=1}^{n-1} t_{i,n-1}^{-\frac{i}{n}}. \tag{4.8}$$

(b) Let $\lambda_n \in \mathbb{C}$, $a_n = (a_{i,n})_{i=1}^n \in (\mathbb{C}^\times)^n$, $\tilde{a}_{n-1} = (\tilde{a}_{i,n-1})_{i=1}^{n-1} \in (\mathbb{C}^\times)^{n-1}$, $b_n = (b_{i,n})_{i=1}^n \in \mathbb{C}^n$, and $\tilde{b}_{n-1} = (\tilde{b}_{i,n-1})_{i=1}^{n-1} \in \mathbb{C}^{n-1}$. Then we obtain a $\mathbb{C}$-algebra homomorphism $\rho_n^B := \rho_n^B(a_n, \tilde{a}_{n-1}, b_n, \tilde{b}_{n-1}, \lambda_n) : U_\varepsilon(B_n) \longrightarrow End(V_n) \otimes End(\widetilde{V}_{n-1}) \otimes U_\varepsilon(B_{n-1})$ such that

$$\begin{aligned}
\rho_n^B(e_{i,n}) &= \{z_{i+1,n} z_{i,n}^{-1}\} x_{i,n} + \{\tilde{z}_{i-1,n-1} \tilde{z}_{i,n-1}^{-1}\} x_{i+1,n}^{-1} x_{i,n} \tilde{x}_{i,n-1} \\
&\quad + x_{i+1,n}^{-1} x_{i,n} \tilde{x}_{i,n-1} \tilde{x}_{i-1,n-1}^{-1} e_{i,n-1} \quad (2 \le i \le n), \\
\rho_n^B(e_{1,n}) &= \{z_{2,n} z_{1,n}^{-\frac{1}{2}}\}_{\varepsilon^1} x_{1,n} + \{\tilde{z}_{1,n-1}^{-1} z_{1,n}^{\frac{1}{2}}\}_{\varepsilon^1} x_{2,n}^{-1} x_{1,n} \tilde{x}_{1,n-1} + x_{2,n}^{-1} \tilde{x}_{1,n-1} e_{1,n-1}, \qquad (4.9) \\
\rho_n^B(t_{i,n}) &= z_{i+1,n} z_{i,n}^{-2} z_{i-1,n} \tilde{z}_{i-2,n-1} \tilde{z}_{i-1,n-1}^{-2} \tilde{z}_{i,n-1} t_{i,n-1} \quad (2 \le i \le n), \\
\rho_n^B(t_{1,n}) &= z_{2,n} z_{1,n}^{-1} \tilde{z}_{1,n-1} t_{1,n-1}, \qquad (4.10) \\
\rho_n^B(f_{i,n}) &= \{z_{i,n} z_{i-1,n}^{-1} \tilde{z}_{i-2,n-1}^{-1} \tilde{z}_{i-1,n-1}^{2} \tilde{z}_{i,n-1}^{-1} t_{i,n-1}^{-1}\} x_{i,n}^{-1} \\
&\quad + \{\tilde{z}_{i-1,n-1} \tilde{z}_{i,n-1}^{-1} t_{i,n-1}^{-1}\} \tilde{x}_{i-1,n-1}^{-1} + f_{i,n-1} \quad (2 \le i \le n), \\
\rho_n^B(f_{1,n}) &= \{z_{1,n}^{\frac{1}{2}} \tilde{z}_{1,n-1}^{-1} t_{1,n-1}^{-1}\}_{\varepsilon^1} x_{1,n}^{-1} + f_{1,n-1}, \qquad (4.11)
\end{aligned}$$

where

$$t_{n,n-1} := \varepsilon^{-\lambda_n} \prod_{i=1}^{n-1} t_{i,n-1}^{-1}. \tag{4.12}$$

(c) Let $\lambda_n \in \mathbb{C}$, $a_n = (a_{i,n})_{i=1}^n \in (\mathbb{C}^\times)^n$, $\tilde{a}_{n-1} = (\tilde{a}_{i,n-1})_{i=1}^{n-1} \in (\mathbb{C}^\times)^{n-1}$, $b_n = (b_{i,n})_{i=1}^n \in \mathbb{C}^n$, and $\tilde{b}_{n-1} = (\tilde{b}_{i,n-1})_{i=1}^{n-1} \in \mathbb{C}^{n-1}$. Then we obtain a $\mathbb{C}$-algebra homomorphism $\rho_n^C := \rho_n^C(a_n, \tilde{a}_{n-1}, b_n, \tilde{b}_{n-1}, \lambda_n) : U_\varepsilon(C_n) \longrightarrow End(V_n) \otimes End(\widetilde{V}_{n-1}) \otimes U_\varepsilon(C_{n-1})$ such that $\rho_n^C(e_{i,n})$, $\rho_n^C(t_{i,n})$, $\rho_n^C(f_{i,n})$ as in (4.9), (4.10), (4.11) if $3 \le i \le n$, and

$$\begin{aligned}
\rho_n^C(e_{2,n}) &= \{z_{3,n}z_{2,n}^{-1}\}x_{2,n} + \{\tilde{z}_{1,n-1}\tilde{z}_{2,n-1}^{-1}\}x_{3,n}^{-1}x_{2,n}\tilde{x}_{2,n-1} + x_{3,n}^{-1}x_{2,n}\tilde{x}_{2,n-1}\tilde{x}_{1,n-1}^{-1}e_{2,n-1}, \\
\rho_n^C(e_{1,n}) &= \{z_{1,n}^2\tilde{z}_{1,n-1}^{-2}\}_{\varepsilon_1}x_{2,n}^{-2}x_{1,n}\tilde{x}_{1,n-1}^2 + \{z_{2,n}\tilde{z}_{1,n-1}^{-1}\}x_{2,n}^{-1}x_{1,n}\tilde{x}_{1,n-1} \\
&\quad + \{z_{2,n}^2 z_{1,n}^{-2}\}_{\varepsilon_1}x_{1,n} + x_{2,n}^{-2}\tilde{x}_{1,n-1}^2 e_{1,n-1}, \qquad (4.13) \\
\rho_n^C(t_{2,n}) &= z_{3,n}z_{2,n}^{-2}z_{1,n}^2\tilde{z}_{1,n-1}^{-2}\tilde{z}_{2,n-1}t_{i,n-1}, \quad \rho_n^C(t_{1,n}) = z_{2,n}^2 z_{1,n}^{-4}\tilde{z}_{1,n-1}^2 t_{1,n-1}, \qquad (4.14) \\
\rho_n^C(f_{2,n}) &= \{z_{2,n}z_{1,n}^{-2}\tilde{z}_{1,n-1}^2\tilde{z}_{2,n-1}^{-1}t_{2,n-1}^{-1}\}x_{2,n}^{-1} + \{\tilde{z}_{1,n-1}\tilde{z}_{2,n-1}^{-1}t_{2,n-1}^{-1}\}\tilde{x}_{1,n-1}^{-1} + f_{2,n-1}, \\
\rho_n^C(f_{1,n}) &= \{z_{1,n}^2\tilde{z}_{1,n-1}^{-2}t_{1,n-1}^{-1}\}_{\varepsilon_1}x_{1,n}^{-1} + f_{1,n-1}, \qquad (4.15)
\end{aligned}$$

$$t_{n,n-1} := \varepsilon^{-\lambda_n} t_{1,n-1}^{-\frac{1}{2}} \prod_{i=2}^{n-1} t_{i,n-1}^{-1} \quad (n \ge 2), \quad t_{1,0} := \varepsilon^{-\lambda_1}. \tag{4.16}$$

(d) Let $\lambda_n \in \mathbb{C}$, $a_n = (a_{i,n})_{i=1}^n \in (\mathbb{C}^\times)^n$, $\tilde{a}_{n-2} = (\tilde{a}_{i,n-2})_{i=1}^{n-2} \in (\mathbb{C}^\times)^{n-2}$, $b_n = (b_{i,n})_{i=1}^n \in \mathbb{C}^n$, and $\tilde{b}_{n-2} = (\tilde{b}_{i,n-2})_{i=1}^{n-2} \in \mathbb{C}^{n-2}$. Then we obtain a $\mathbb{C}$-algebra homomorphism $\rho_n^D := \rho_n^D(a_n, \tilde{a}_{n-2}, b_n, \tilde{b}_{n-2}, \lambda_n) : U_\varepsilon(D_n) \longrightarrow End(V_n) \otimes End(\widetilde{V}_{n-2}) \otimes U_\varepsilon(D_{n-1})$ such that $\rho_n^D(e_{i,n})$, $\rho_n^D(t_{i,n})$, $\rho_n^D(f_{i,n})$ as in (4.9), (4.10), (4.11) if $4 \le i \le n$ (replace $\tilde{x}_{i,n-1}$ to $\tilde{x}_{i-1,n-2}$ and $\tilde{z}_{i,n-1}$ to $\tilde{z}_{i-1,n-2}$). Moreover,

$$\begin{aligned}
\rho_n^D(e_{3,n}) &= \{z_{4,n}z_{3,n}^{-1}\}x_{3,n} + \{\tilde{z}_{1,n-2}\tilde{z}_{2,n-2}^{-1}\}x_{4,n}^{-1}x_{3,n}\tilde{x}_{2,n-2} + x_{4,n}^{-1}x_{3,n}\tilde{x}_{2,n-2}\tilde{x}_{1,n-2}^{-1}e_{3,n-1}, \\
\rho_n^D(e_{2,n}) &= \{z_{3,n}z_{2,n}^{-1}\}x_{2,n} + \{z_{1,n}\tilde{z}_{1,n-2}^{-1}\}\tilde{x}_{1,n-2}x_{3,n}^{-1}x_{2,n} + x_{3,n}^{-1}x_{2,n}x_{1,n}^{-1}\tilde{x}_{1,n-2}e_{2,n-1}, \\
\rho_n^D(e_{1,n}) &= \{z_{3,n}z_{1,n}^{-1}\}x_{1,n} + \{z_{2,n}\tilde{z}_{1,n-2}^{-1}\}\tilde{x}_{1,n-2}x_{3,n}^{-1}x_{1,n} + x_{3,n}^{-1}x_{1,n}x_{2,n}^{-1}\tilde{x}_{1,n-2}e_{2,n-1}, \\
& \qquad (4.17) \\
\rho_n^D(t_{3,n}) &= z_{4,n}z_{3,n}^{-2}z_{2,n}z_{1,n}\tilde{z}_{1,n-2}^{-2}\tilde{z}_{2,n-2}t_{i,n-1}, \\
\rho_n^D(t_{2,n}) &= z_{3,n}z_{2,n}^{-2}\tilde{z}_{1,n-2}t_{2,n-1}, \quad \rho_n^D(t_{1,n}) = z_{3,n}z_{1,n}^{-2}\tilde{z}_{1,n-2}t_{1,n-1}, \qquad (4.18) \\
\rho_n^D(f_{3,n}) &= \{z_{3,n}z_{2,n}^{-1}z_{1,n}^{-1}\tilde{z}_{1,n-2}^2\tilde{z}_{2,n-2}^{-1}t_{3,n-1}^{-1}\}x_{3,n}^{-1} + \{\tilde{z}_{1,n-2}\tilde{z}_{2,n-2}^{-1}t_{3,n-1}^{-1}\}\tilde{x}_{1,n-2}^{-1} + f_{3,n-1}, \\
\rho_n^D(f_{2,n}) &= \{z_{2,n}\tilde{z}_{1,n-2}^{-1}t_{2,n-1}^{-1}\}x_{2,n}^{-1} + f_{2,n-1}, \\
\rho_n^D(f_{1,n}) &= \{z_{1,n}\tilde{z}_{1,n-2}^{-1}t_{1,n-1}^{-1}\}x_{1,n}^{-1} + f_{1,n-1}, \qquad (4.19)
\end{aligned}$$

$$t_{n,n-1} := \varepsilon^{-\lambda_n} t_{1,n-1}^{-\frac{1}{2}} t_{2,n-1}^{-\frac{1}{2}} \prod_{i=3}^{n-1} t_{i,n-1}^{-1}, \quad (n \ge 3), \quad t_{2,1} := \varepsilon^{-\lambda_2} t_{1,1}^{-\frac{1}{2}}, \quad t_{1,0} := \varepsilon^{-\lambda_1}. \tag{4.20}$$

(e) Let $\lambda_2 \in \mathbb{C}$, $a_2 = (a_{i,2})_{i=1}^5 \in (\mathbb{C}^\times)^5$, and $b_2 = (b_{i,2})_{i=1}^5 \in \mathbb{C}^5$. Then we obtain a

$\mathbb{C}$-algebra homomorphism $\rho^G := \rho^G(a_2, b_2, \lambda_2) : U_\varepsilon(G_2) \longrightarrow End(V_5) \otimes U_\varepsilon(A_1)$ such that

$$\begin{aligned}
\rho^G(e_{1,2}) &= \{z_{3,2}^3 z_{4,2}^{-2}\} x_{1,2}^{-1} x_{2,2}^{-1} x_{3,2} x_{4,2}^2 + \{z_{4,2} z_{5,2}^{-3}\} x_{1,2}^{-1} x_{2,2}^{-1} x_{4,2}^2 x_{5,2} + \{z_{2,2}^2 z_{3,2}^{-3}\} x_{1,2}^{-1} x_{2,2} x_{3,2} \\
&\quad + [2]\{z_{2,2} z_{4,2}^{-1}\} x_{1,2}^{-1} x_{3,2} x_{4,2} + \{z_{1,2}^3 z_{2,2}^{-1}\} x_{2,2} + x_{1,2}^{-1} x_{2,2}^{-1} x_{4,2} x_{5,2} e_{1,1}, \\
\rho^G(e_{2,2}) &= \{z_{1,2}^{-3}\}_{\varepsilon^3} x_{1,2}, \qquad (4.21) \\
\rho^G(t_{1,2}) &= z_{1,2}^3 z_{3,2}^3 z_{5,2}^3 z_{2,2}^{-2} z_{4,2}^{-2} t_{1,1}, \quad \rho^G(t_{2,2}) = z_{1,2}^{-6} z_{3,2}^{-6} z_{5,2}^{-6} z_{2,2}^3 z_{4,2}^3 \varepsilon^{\lambda_2} t_{1,1}^{-\frac{3}{2}}, \qquad (4.22) \\
\rho^G(f_{1,2}) &= \{z_{2,2} z_{3,2}^{-3} z_{5,2}^{-3} z_{4,2}^2 t_{1,1}^{-1}\} x_{2,2}^{-1} + \{z_{4,2} z_{5,2}^{-3} t_{1,1}^{-1}\} x_{4,2}^{-1} + f_{1,1}, \\
\rho^G(f_{2,2}) &= \{z_{1,2}^3 z_{3,2}^6 z_{5,2}^6 z_{2,2}^{-3} z_{4,2}^{-3} \varepsilon^{-\lambda_2} t_{1,1}^{\frac{3}{2}}\}_{\varepsilon^3} x_{1,2}^{-1} + \{z_{3,2}^3 z_{5,2}^6 z_{4,2}^{-3} \varepsilon^{-\lambda_2} t_{1,1}^{\frac{3}{2}}\}_{\varepsilon^3} x_{3,2}^{-1} \\
&\quad + \{z_{5,2}^3 \varepsilon^{-\lambda_2} t_{1,1}^{\frac{3}{2}}\}_{\varepsilon^3} x_{5,2}^{-1}. \qquad (4.23)
\end{aligned}$$

Here, $x_{i,j}^{\pm 1} := 0$, $z_{i,j}^{\pm 1} := 1$ if the index (i,j) is out of range, $e_{0,n} := f_{n,n-1} := 0$, $U_\varepsilon(\mathfrak{g}_0) := \mathbb{C}$, $V_0 := \mathbb{C}$, and $\widetilde{V}_j$, $\tilde{x}_{i,j}$, $\tilde{z}_{i,j}$ are a copy of $V_j, x_{i,j}, z_{i,j}$.

By using these homomorphisms, we obtain l^N-dimensional $U_\varepsilon(\mathfrak{g}_n)$-modules having $\dim\mathfrak{g}_n$-parameters. We call those modules the Schnizer modules.

Remark 4.2. *The actions of $e_{i,n}, t_{i,n}, f_{i,n}$ in [10] are slightly different from the one of Theorem 4.1. Because we use a $U_\varepsilon(\mathfrak{g}_n)$-automorphism ω such that $(\omega(e_{i,n}), \omega(t_{i,n}), \omega(f_{i,n})) = (f_{i,n}, t_{i,n}^{-1}, e_{i,n})$.*

Now, we introduce the following fact to use later. If $\mathfrak{g}_n = A_n$ $(n \geq 2)$, by (4.6), (4.8), we obtain

$$\begin{aligned}
\rho_{n-1}^A(t_{n,n-1}) &= \varepsilon^{-\lambda_n} \prod_{i=1}^{n-1} \rho_{n-1}^A(t_{i,n-1}^{-\frac{i}{n}}) = \varepsilon^{-\lambda_n} \prod_{i=1}^{n-1} (z_{i-1,n-1} z_{i,n-1}^{-2} z_{i+1,n-1} t_{i,n-2})^{-\frac{i}{n}} \\
&= \varepsilon^{-\lambda_n} z_{n-1,n-1} \prod_{i=1}^{n-1} t_{i,n-2}^{-\frac{i}{n}} = \varepsilon^{-\lambda_n} z_{n-1,n-1} t_{n-1,n-2}^{-\frac{n-1}{n}} \prod_{i=1}^{n-2} t_{i,n-2}^{-\frac{i}{n}} \\
&= \varepsilon^{-\lambda_n} z_{n-1,n-1} (\varepsilon^{-\lambda_{n-1}} \prod_{i=1}^{n-2} t_{i,n-2}^{-\frac{i}{n-1}})^{-\frac{n-1}{n}} \prod_{i=1}^{n-2} t_{i,n-2}^{-\frac{i}{n}} \\
&= \varepsilon^{-\lambda_n + \frac{n-1}{n}\lambda_{n-1}} z_{n-1,n-1}. \qquad (4.24)
\end{aligned}$$

Similarly, by (4.10), (4.12), (4.14), (4.16), (4.18), (4.20), we obtain

$$\begin{aligned}
\rho_{n-1}^B(t_{n,n-1}) &= \varepsilon^{-\lambda_n + \lambda_{n-1}} z_{n-1,n-1} \tilde{z}_{n-2,n-2} \quad (n \geq 2), \qquad (4.25) \\
\rho_{n-1}^C(t_{n,n-1}) &= \varepsilon^{-\lambda_n + \lambda_{n-1}} z_{n-1,n-1} \tilde{z}_{n-2,n-2} \quad (n \geq 3), \\
\rho_1^C(t_{2,1}) &= \varepsilon^{-\lambda_2 + \frac{1}{2}\lambda_1} z_{1,1}^2, \qquad (4.26) \\
\rho_{n-1}^D(t_{n,n-1}) &= \varepsilon^{-\lambda_n + \lambda_{n-1}} z_{n-1,n-1} \tilde{z}_{n-3,n-3} \quad (n \geq 4), \\
\rho_2^D(t_{3,2}) &= \varepsilon^{-\lambda_3 + \frac{1}{2}\lambda_2 + \frac{1}{4}} z_{1,1}^{\frac{1}{2}} z_{1,2} \tilde{z}_{2,2}, \quad \rho_1^D(t_{2,1}) = \varepsilon^{-\lambda_2 + \frac{1}{2}\lambda_1} z_{1,1}, \qquad (4.27)
\end{aligned}$$

$\rho_{n-1}^{\mathfrak{g}}(t_{1,0}) = \varepsilon^{-\lambda_1}$ for $\mathfrak{g} = A, B, C$ or D.

5 Construction of $L_\varepsilon^{\mathbf{nil}}(\lambda)$ (Type G-case)

In this section, we construct all finite dimensional irreducible nilpotent $U_\varepsilon(G_2)$-modules of type 1 by using the Schnizer-homomorphisms in Theorem 4.1(e).

We set

$$
\begin{aligned}
&a^{(0)}_{1,1} := a^{(0)}_{i,2} := 1 \quad (1 \le i \le 5),\\
&b^{(0)}_{1,1} := b^{(0)}_{1,2} := 1, \quad b^{(0)}_{2,2} := 4, \quad b^{(0)}_{4,2} := 5, \quad b^{(0)}_{3,2} := 3, \quad b^{(0)}_{5,2} := 2,\\
&a^{(0)}_{2} := (a^{(0)}_{i,2})_{i=1}^5, \quad b^{(0)}_{2} := (b^{(0)}_{i,2})_{i=1}^5.
\end{aligned} \tag{5.1}
$$

For $\lambda \in \mathbb{C}$, we set

$$
\begin{aligned}
&\rho^A_1(\lambda) := \rho^A_1(a^{(0)}_{1,1}, b^{(0)}_{1,1}, \lambda) : U_\varepsilon(A_1) \longrightarrow \mathrm{End}(\mathbb{C}),\\
&\rho^G(\lambda) := \rho^G(a^{(0)}_{2}, b^{(0)}_{2}, \lambda) : U_\varepsilon(G_2) \longrightarrow \mathrm{End}(V_5) \otimes U_\varepsilon(A_1),
\end{aligned} \tag{5.2}
$$

(see Theorem 4.1(a), (e)). For $\lambda_1, \lambda_2 \in \mathbb{C}$, we define

$$
\phi_{1,2} := \phi_{1,2}(\lambda_1,\lambda_2) := \rho^{A_1}_1(\nu_1^{(\lambda_1,\lambda_2)}) \circ \rho^G(\nu_2^{(\lambda_1,\lambda_2)}) : U_\varepsilon(G_2) \longrightarrow \mathrm{End}(V_5 \otimes V_1), \tag{5.3}
$$

where

$$
\nu_1^{(\lambda_1,\lambda_2)} := \lambda_1 + 2, \quad \nu_2^{(\lambda_1,\lambda_2)} := \frac{3}{2}\lambda_1 + 3\lambda_2 + 9.
$$

We denote the $U_\varepsilon(G_2)$-modules associated with $(\phi_{1,2}(\lambda_1,\lambda_2), V_5 \otimes V_1)$ by $V_{1,2}(\lambda_1,\lambda_2)$. For $m_{1,1} \in \mathbb{Z}_l$, $m_5 = (m_{i,5})_{i=1}^5 \in \mathbb{Z}_l^5$, we set

$$
v_{1,2}(m_5, m_{1,1}) := v_5(m_5) \otimes v_1(m_{1,1}), \quad v^{\mathbf{0}}_{1,2} := v_{1,2}(0,\cdots,0) \in V_{1,2}(\lambda_1,\lambda_2).
$$

We define $y_{i,2}$, $y_{1,1} \in \mathrm{End}(V_{1,2}(\lambda_1,\lambda_2))$ $(1 \le i \le 5)$: for $v = v_{1,2}(m_5, m_{1,1})$,

$$
\begin{aligned}
y_{1,2}v &:= [m_{1,2} + 2m_{3,2} + 2m_{5,2} - m_{2,2} - m_{4,2} - m_{1,1} - \lambda_2]_{\varepsilon_2} v_{1,2}(m_5 + \epsilon_{1,2}, m_{1,1}),\\
y_{2,2}v &:= [m_{2,2} - 3m_{3,2} - 3m_{5,2} + 2m_{4,2} + 2m_{1,1} - \lambda_1] v_{1,2}(m_5 + \epsilon_{2,2}, m_{1,1}),\\
y_{3,2}v &:= [m_{3,2} + 2m_{5,2} - m_{4,2} - m_{1,1} - \lambda_2]_{\varepsilon_2} v_{1,2}(m_5 + \epsilon_{3,2}, m_{1,1}),\\
y_{4,2}v &:= [m_{4,2} - 3m_{5,2} + 2m_{1,1} - \lambda_1] v_{1,2}(m_5 + \epsilon_{4,2}, m_{1,1}),\\
y_{5,2}v &:= [m_{5,2} - m_{1,1} - \lambda_2]_{\varepsilon_2} v_{1,2}(m_5 + \epsilon_{5,2}, m_{1,1}),\\
y_{1,1}v &:= [m_{1,1} - \lambda_1] v_{1,2}(m_5, m_{1,1} + \epsilon_{1,1}).
\end{aligned} \tag{5.4}
$$

Then, by Theorem 4.1(a), (e), (4.2), (4.4), (5.1), we have

$$
\begin{aligned}
e_{1,2}v = {} & [3m_{3,2} - 2m_{4,2}] v_{1,2}(m_{1,2}+1, m_{2,2}+1, m_{3,2}-1, m_{4,2}-2, m_{5,2}, m_{1,1})\\
&+[m_{4,2} - 3m_{5,2}] v_{1,2}(m_{1,2}+1, m_{2,2}+1, m_{3,2}, m_{4,2}-2, m_{5,2}-1, m_{1,1})\\
&+[2m_{2,2} - 3m_{3,2}] v_{1,2}(m_{1,2}+1, m_{2,2}-1, m_{3,2}-1, m_{4,2}, m_{5,2}, m_{1,1})\\
&+[2][m_{2,2} - m_{4,2}] v_{1,2}(m_{1,2}+1, m_{2,2}, m_{3,2}-1, m_{4,2}-1, m_{5,2}, m_{1,1})\\
&+[3m_{1,2} - m_{2,2}] v_{1,2}(m_{1,2}, m_{2,2}-1, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1})\\
&+[-m_{1,1}] v_{1,2}(m_{1,2}+1, m_{2,2}+1, m_{3,2}, m_{4,2}-1, m_{5,2}-1, m_{1,1}-1),
\end{aligned} \tag{5.5}
$$

$$
e_{2,2}v = [-m_{1,2}]_{\varepsilon_2} v_{1,2}(m_{1,2}-1, m_{2,2}, m_{3,2}, m_{4,2}, m_{5,2}, m_{1,1}), \tag{5.6}
$$

$$
t_{1,2}v = \varepsilon^{3m_{1,2}+3m_{3,2}+3m_{5,2}-2m_{2,2}-2m_{4,2}-2m_{1,1}+\lambda_1} v_{1,2}(m_5, m_{1,1}), \tag{5.7}
$$

$$
t_{2,2}v = \varepsilon_2^{-2m_{1,2}-2m_{3,2}-2m_{5,2}+m_{2,2}+m_{4,2}+m_{1,1}+\lambda_2} v_{1,2}(m_5, m_{1,1}), \tag{5.8}
$$

$$
f_{1,2}v = (y_{2,2} + y_{4,2} + y_{1,1}) v_{1,2}(m_5, m_{1,1}), \tag{5.9}
$$

$$
f_{2,2}v = (y_{1,2} + y_{3,2} + y_{5,2}) v_{1,2}(m_5, m_{1,1}). \tag{5.10}
$$

Let $P(V_{1,2}(\lambda_1,\lambda_2))$ as in Definition 3.3 (i).

Proposition 5.1. *For all λ_1, $\lambda_2 \in \mathbb{C}$, we obtain $P(V_{1,2}(\lambda_1,\lambda_2)) = \mathbb{C}v^{\mathbf{0}}_{1,2}$.*

Proof. Since the actions of $e_{1,2}$, $e_{2,2}$ on $V_{1,2}(\lambda_1,\lambda_2)$ do not depend on λ_1, λ_2, we simply denote $V_{1,2}(\lambda_1,\lambda_2)$ by $V_{1,2}$. By (5.5), (5.6), obviously, $\mathbb{C}v^{\mathbf{0}}_{1,2}\subset P(V_{1,2})$. So we shall prove $P(V_{1,2})\subset \mathbb{C}v^{\mathbf{0}}_{1,2}$. Let

$$v=\sum_{m_5\in\mathbb{Z}_l^5,m_{1,1}\in\mathbb{Z}_l} c(m_5,m_{1,1})v_{1,2}(m_5,m_{1,1})\in V_{1,2},$$

where $c(m_5,m_{1,1})\in\mathbb{C}$, and we assume that $e_{1,2}v=e_{2,2}v=0$. By (5.6), we get

$$0=e_{2,2}v=\sum_{m_5\in\mathbb{Z}_l^5,m_{1,1}\in\mathbb{Z}_l} c(m_5,m_{1,1})[-m_{1,2}]v_{1,2}(m_{1,2}-1,m_{2,2},m_{3,2},m_{4,2},m_{5,2},m_{1,1}).$$

Hence, we obtain $c(m_5,m_{1,1})=0$ if $m_{1,2}\neq 0$. So, by (5.5), we have

$$\begin{aligned}0=e_{1,2}v \quad = \quad & \sum_{m_{2,2},m_{3,2},m_{4,2},m_{5,2},m_{1,1}\in\mathbb{Z}_l} c(0,m_{2,2},m_{3,2},m_{4,2},m_{5,2},m_{1,1})\\
&\{[3m_{3,2}-2m_{4,2}]v_{1,2}(1,m_{2,2}+1,m_{3,2}-1,m_{4,2}-2,m_{5,2},m_{1,1})\\
&+[m_{4,2}-3m_{5,2}]v_{1,2}(1,m_{2,2}+1,m_{3,2},m_{4,2}-2,m_{5,2}-1,m_{1,1})\\
&+[2m_{2,2}-3m_{3,2}]v_{1,2}(1,m_{2,2}-1,m_{3,2}-1,m_{4,2},m_{5,2},m_{1,1})\\
&+[2][m_{2,2}-m_{4,2}]v_{1,2}(1,m_{2,2},m_{3,2}-1,m_{4,2}-1,m_{5,2},m_{1,1})\\
&+[-m_{2,2}]v_{1,2}(0,m_{2,2}-1,m_{3,2},m_{4,2},m_{5,2},m_{1,1})\\
&+[-m_{1,1}]v_{1,2}(1,m_{2,2}+1,m_{3,2},m_{4,2}-1,m_{5,2}-1,m_{1,1}-1)\}.\end{aligned}$$

Since the $(1,2)$-component of $(0,m_{2,2}-1,m_{3,2},m_{4,2},m_{5,2},m_{1,1})$ is 0 and the one of other vectors is 1, by the linearly independence, $c(m_5,m_{1,1})=0$ if $m_{2,2}\neq 0$. Therefore we obtain

$$\begin{aligned}0=e_{1,2}v \quad = \quad & \sum_{m_{3,2},m_{4,2},m_{5,2},m_{1,1}\in\mathbb{Z}_l} c(0,0,m_{3,2},m_{4,2},m_{5,2},m_{1,1})\\
&\{[3m_{3,2}-2m_{4,2}]v_{1,2}(1,1,m_{3,2}-1,m_{4,2}-2,m_{5,2},m_{1,1})\\
&+[m_{4,2}-3m_{5,2}]v_{1,2}(1,1,m_{3,2},m_{4,2}-2,m_{5,2}-1,m_{1,1})\\
&+[-3m_{3,2}]v_{1,2}(1,-1,m_{3,2}-1,m_{4,2},m_{5,2},m_{1,1})\\
&+[2][-m_{4,2}]v_{1,2}(1,0,m_{3,2}-1,m_{4,2}-1,m_{5,2},m_{1,1})\\
&+[-m_{1,1}]v_{1,2}(1,1,m_{3,2},m_{4,2}-1,m_{5,2}-1,m_{1,1}-1)\}.\end{aligned}$$

Since the $(2,2)$-component of $(1,-1,m_{3,2}-1,m_{4,2},m_{5,2},m_{1,1})$ (resp. $(1,0,m_{3,2}-1,m_{4,2}-1,m_{5,2},m_{1,1})$) is -1 (resp. 0) and the one of other vectors is 1, we get $c(m_5,m_{1,1})=0$ if $m_{3,2}\neq 0$ or $m_{4,2}=0$. Hence we have

$$\begin{aligned}0=e_{1,2}v \quad = \quad & \sum_{m_{5,2},m_{1,1}\in\mathbb{Z}_l} c(0,0,0,0,m_{5,2},m_{1,1})\{[-3m_{5,2}]v_{1,2}(1,1,0,-2,m_{5,2}-1,m_{1,1})\\
&+[-m_{1,1}]v_{1,2}(1,1,0,-1,m_{5,2}-1,m_{1,1}-1)\}.\end{aligned}$$

Since the $(4,2)$-component of $(1,1,0,-2,m_{5,2}-1,m_{1,1})$ is -2 and the one of $(1,1,0,-1,m_{5,2}-1,m_{1,1}-1)$ is -1, we obtain $c(m_5,m_{1,1})=0$ if $m_{5,2}\neq 0$ or $m_{1,1}=0$. It amount to $v=c(0,\cdots,0)v^{\mathbf{0}}_{1,2}\in\mathbb{C}v^{\mathbf{0}}_{1,2}$. □

Let $y_{i,2}$ $(1\le i\le 5)$, $y_{1,1}$ as in (5.4) and we set $Y_{1,2}:=\{y_{i,2},y_{1,1}\,|\,1\le i\le 5\}$. Let $p_{\mathbf{0}}:V_{1,2}(\lambda)\longrightarrow \mathbb{C}v^{\mathbf{0}}_{1,2}$ be the projection.

Lemma 5.2. *Let $\lambda_1,\ \lambda_2 \in \mathbb{Z}$.*

(a) For all $r \in \mathbb{N}$, $g_1, \cdots, g_r \in Y_{1,2}$, we have

$$p_{\boldsymbol{0}}(g_1 \cdots g_r v_{1,2}^{\boldsymbol{0}}) = 0 \quad \text{in } V_{1,2}(\lambda_1, \lambda_2).$$

(b) For all $r \in \mathbb{N}$, $i_1, \cdots, i_r \in \{1, 2\}$, we have

$$p_{\boldsymbol{0}}(f_{i_1,2} \cdots f_{i_r,2} v_{1,2}^{\boldsymbol{0}}) = 0 \quad \text{in } V_{1,2}(\lambda_1, \lambda_2).$$

Proof. If we can prove (a), then we obtain (b) by (5.9), (5.10). So we shall prove (a). Now we fix $r \in \mathbb{N}$, $g_1, \cdots, g_r \in Y_{1,2}$ and set $g := g_1 \cdots g_r$. For $y \in Y_{1,2}$, we set

$$s(y) := \#\{1 \le i \le r \,|\, g_i = y\} \ge 0, \quad m_g := \sum_{i=1}^{5} s(y_{i,2})\epsilon_{i,2} + s(y_{1,1})\epsilon_{1,1},$$

$$W_g := \bigoplus_{s=1}^{r} \mathbb{C}(g_s g_{s+1} \cdots g_r v_{1,2}^{\mathbf{0}}) \subset V_{1,2}(\lambda_1, \lambda_2).$$

Then, $gv_{1,2}^{\mathbf{0}} \in \mathbb{C}v_{1,2}(m_g)$ by (5.4), (5.9), (5.10). Since $\sum_{i=1}^{5} s(y_{i,2}) + s(y_{1,1}) = r > 0$, there exists a $1 \le i \le 5$ such that $s(y_{i,2}) > 0$ or $s(y_{1,1}) > 0$.

Case 1) $s(y_{1,1}) > 0$: For $1 \le r' \le r$, let $m^{(r')} \in \mathbb{Z}_l^6$ such that $g_s g_{s+1} \cdots g_r \in \mathbb{C}v_{1,2}(m^{(r')})$. Let $1 \le r_1 \le r$ such that $g_{r_1} = y_{1,1}$ and $g_{r_1+1}, \cdots, g_r \ne y_{1,1}$. Then, by (5.4), $m_{1,1}^{(r_1+1)} = 0$. Hence, by the definition of $y_{1,1}$ in (5.4), we get

$$g_{r_1} g_{r_1+1} \cdots g_r v_{1,2}^{\mathbf{0}} \in \mathbb{C}[-\lambda_1] v_{1,2}(m^{(r_1+1)} + \epsilon_{1,1}).$$

Similarly, for $1 \le r_2 < r_1$ such that $g_{r_2} = y_{1,1}$ and $g_{r_2+1}, \cdots, g_{r_1-1} \ne y_{1,1}$, we have

$$g_{r_2} g_{r_2+1} \cdots g_r v_{1,2}^{\mathbf{0}} \in \mathbb{C}[-\lambda_1 + 1][-\lambda_1] v_{1,2}(m^{(r_2+1)} + 2\epsilon_{1,1}).$$

By repeating this, we obtain

$$gv_{1,2}^{\mathbf{0}} \in \mathbb{C}[-\lambda_1 + s(y_{1,1}) - 1] \cdots [-\lambda_1 + 1][-\lambda_1] v_{1,2}(m_g).$$

Since $\lambda_1 \in \mathbb{Z}$ and $[l] = 0$, if $s(y_{1,1}) \ge l$, then $[-\lambda_1 + s(y_{1,1} - 1] \cdots [-\lambda_1 + 1][-\lambda_1] = 0$. On the other hand, if $0 < s(y_{1,1}) < l$, then $p_{\mathbf{0}}(v_{1,2}(m_g)) = 0$. Therefore, we obtain $p_{\mathbf{0}}(gv_{1,2}^{\mathbf{0}}) = 0$.

Case 2) $s(y_{1,1}) = 0$ and $s(y_{5,2}) > 0$: Since $s(y_{1,1}) = 0$, for all $1 \le r' \le r$, $m_{1,1}^{(r')} = 0$. Hence, we get

$$y_{5,2} v_{1,2}(m_5, m_{1,1}) = [m_{5,2} - \lambda_2]_{\varepsilon_2} v_{1,2}(m_5 + \epsilon_{5,2}, m_{1,1}) \quad \text{in } W_g.$$

Thus, by the similar way to the proof of Case 1, we obtain $p_{\mathbf{0}}(gv_{1,2}^{\mathbf{0}}) = 0$.

Case 3) There exists a $1 \le i \le 4$ such that $s(y_{1,1}) = s(y_{5,2}) = \cdots = s(y_{i+1,2}) = 0$ and $s(y_{i,2}) > 0$: In this case , for all $1 \le r' \le r$, $m_{1,1}^{(r')} = m_{5,2}^{(r')} = \cdots = m_{i+1,2}^{(r')} = 0$. Hence we have

$$y_{i,2} v_{1,2}(m_5, m_{1,1}) = [m_{i,2} - \lambda_{\tilde{i}}]_{\varepsilon_{\tilde{i}}} v_{1,2}(m_5 + \epsilon_{i,2}, m_{1,1}) \quad \text{in } W_g,$$

where $\tilde{i} := 1$ if $i = 2, 4$ and $\tilde{i} := 2$ if $i = 1, 3$. Therefore, by the similar way to the proof of Case 1, we obtain $p_{\mathbf{0}}(gv_{1,2}^{\mathbf{0}}) = 0$.

By Case1–3, we obtain $p_{\mathbf{0}}(gv_{1,2}^{\mathbf{0}}) = 0$. $\square$

Lemma 5.3. *For all $\lambda_1,\ \lambda_2 \in \mathbb{Z}$, $\alpha \in \Delta_+$, we have $f_{\alpha,2}^{l} v_{1,2}^{\boldsymbol{0}} = 0$ in $V_{k,n}(\lambda)$.*

Proof. By Lemma 5.2(b) and Proposition 2.4, we obtain $p_{\mathbf{0}}(f_{\alpha,2}^l v_{1,2}^{\mathbf{0}}) = 0$. On the other hand, by Proposition 2.3, 5.1,

$$e_{i,2}f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = f_{\alpha,2}^l e_{i,2} v_{1,2}^{\mathbf{0}} = 0 \quad (i = 1, 2).$$

Hence, by Proposition 5.1, we get $f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} \in \mathbb{C}v_{1,2}^{\mathbf{0}}$. Therefore, we obtain

$$f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = p_{\mathbf{0}}(f_{\alpha,2}^l v_{1,2}^{\mathbf{0}}) = 0.$$

□

Now, we construct nilpotent $U_\varepsilon(G_2)$-modules (see §3). For $\lambda_1,\ \lambda_2 \in \mathbb{C}$, let $L_{1,2}(\lambda_1, \lambda_2)$ be the $U_\varepsilon(G_2)$-submodule of $V_{1,2}(\lambda_1, \lambda_2)$ generated by $v_{1,2}^{\mathbf{0}}$.

Theorem 5.4. *For any $\lambda_1,\ \lambda_2 \in \mathbb{Z}_l$, $L_{1,2}(\lambda_1, \lambda_2)$ is isomorphic to $L_\varepsilon^{nil}(\lambda_1, \lambda_2)$ as $U_\varepsilon(G_2)$-module.*

Proof. By Proposition 5.1, $e_{1,2}v_{1,2}^{\mathbf{0}} = e_{2,2}v_{1,2}^{\mathbf{0}} = 0$. Moreover, by (5.7), (5.8),

$$t_{i,2}v_{1,2}^{\mathbf{0}} = \varepsilon_i^{\lambda_i} v_{1,2}^{\mathbf{0}} \quad (i = 1, 2).$$

So $L_{1,2}(\lambda_1, \lambda_2)$ is a finite dimensional highest weight $U_\varepsilon(G_2)$-module with highest weight (λ_1, λ_2). On the other hand, by Lemma 5.3, $f_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = 0$ for all $\alpha \in \Delta_+$. Moreover, by Proposition 2.4, 5.1, we have $e_{\alpha,2}^l v_{1,2}^{\mathbf{0}} = 0$ for all $\alpha \in \Delta_+$. Hence, by Proposition 2.3, $e_{\alpha,2}^l = f_{\alpha,2}^l = 0$ on $L_{1,2}(\lambda_1, \lambda_2)$ for all $\alpha \in \Delta_+$. Thus $L_{1,2}(\lambda_1, \lambda_2)$ is a nilpotent $U_\varepsilon(G_2)$-module. Therefore, by Proposition 5.1 and Proposition 3.5, we obtain this theorem. □

If $\lambda_1 = 0$, then we can construct $L_\varepsilon^{\mathrm{nil}}(\lambda_1, \lambda_2)$ more easily. For $\lambda \in \mathbb{C}$, let $\rho^G(\lambda)$ as in (5.2). For $m \in \mathbb{Z}_+$, let $(\pi_m, \mathbb{C})$ be the trivial representation of $U_\varepsilon(\mathfrak{g}_m)$, that is,

$$\pi_m(e_{i,m}) = \pi_m(f_{i,m}) = 0, \quad \pi_m(t_{i,m}) = 1 \quad (1 \le i \le m), \tag{5.11}$$

where $e_{i,0} := f_{i,0} := 0$, $t_{i,0} := 1$, $U_\varepsilon(\mathfrak{g}_0) := \mathbb{C}$. For $\lambda_2 \in \mathbb{C}$, we define

$$\phi_{2,2} := \phi_{2,2}(\lambda_2) := \pi_1 \circ \rho^G(\nu_2^{\lambda_2}) : U_\varepsilon(G_2) \longrightarrow \mathrm{End}(V_5),$$

where $\nu_2^{\lambda_2} := 3\lambda_2 + 9$. We denote the $U_\varepsilon(G_2)$-module associated with $(\phi_{2,2}(\lambda_2), V_5)$ by $V_{2,2}(\lambda_2)$. Let $L_{2,2}(\lambda_2)$ be the $U_\varepsilon(G_2)$-submodule of $V_{2,2}(\lambda_2)$ generated by $v_{2,2}^{\mathbf{0}} := v_5(0, \cdots, 0)$. Then, by the similar way to the proof of Proposition 5.1, Lemma 5.2, 5.3, and Theorem 5.4, we obtain the following proposition.

Proposition 5.5. *For any $\lambda_2 \in \mathbb{Z}_l$, $L_{2,2}(\lambda_2)$ is isomorphic to $L_l^{nil}(0, \lambda_2)$ as $U_\varepsilon(G_2)$-module. In particular, for any $\lambda_2 \in \mathbb{C}$, we have $P(V_{2,2}(\lambda_2)) = \mathbb{C}v_{2,2}^{\mathbf{0}}$.*

6 Inductive construction of $L_\varepsilon^{\mathbf{nil}}(\lambda)$ (Type B-case)

In this section, we construct all finite dimensional irreducible nilpotent $U_\varepsilon(B_n)$-modules of type 1 inductively by using the Schnizer-homomorphisms of Theorem 4.1(b).

We set $a_n^{(0)} = (a_{i,n}^{(0)})_{i=1}^n$, $\tilde{a}_{n-1}^{(0)} = (\tilde{a}_{i,n-1}^{(0)})_{i=1}^{n-1}$, $b_n^{(0)} = (b_{i,n}^{(0)})_{i=1}^n$, $\tilde{b}_{n-1}^{(0)} = (\tilde{b}_{i,n-1}^{(0)})_{i=1}^{n-1} \in \mathbb{C}^n$ by

$$a_{i,n}^{(0)} := \tilde{a}_{i,n}^{(0)} := 1, \quad b_{i,n}^{(0)} := n - i + 1\,(i \ne 1), \quad b_{1,n}^{(0)} := 2n - 1, \quad \tilde{b}_{i,n-1}^{(0)} := i + n - 2. \tag{6.1}$$

We fix $k \in I$. For $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{C}^{n-k+1}$, we define $\nu^\lambda = (\nu_k^\lambda, \cdots, \nu_n^\lambda) \in \mathbb{C}^{n-k+1}$ by

$$\nu_i^\lambda := -2i + 1 - \sum_{j=k}^{i} \lambda_j \quad (k \geq 2), \quad \nu_i^\lambda := -2i + 1 - \frac{1}{2}\lambda_1 - \sum_{j=2}^{i} \lambda_j \quad (k = 1),$$

where $k \leq i \leq n$. For $\lambda \in \mathbb{C}$, we define $\rho_n^B(\lambda) := \rho_n^B(a_n^{(0)}, \tilde{a}_{n-1}^{(0)}, b_n^{(0)}, \tilde{b}_{n-1}^{(0)}, \lambda) : U_\varepsilon(B_n) \longrightarrow \mathrm{End}(V_n \otimes \widetilde{V}_{n-1}) \otimes U_\varepsilon(B_{n-1})$ (see Theorem 4.1(b)), and let $(\pi_{k-1}, \mathbb{C})$ be as in (5.11). We set

$$V_{k,n} := \bigotimes_{j=k}^{n} (V_j \otimes \widetilde{V}_{j-1}).$$

For $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{C}^{n-k+1}$, we define a $U_\varepsilon(B_n)$-representation $\phi_{k,n} := \phi_{k,n}(\lambda) : U_\varepsilon(B_n) \longrightarrow \mathrm{End}(V_{k,n})$ by

$$\phi_{k,n}(\lambda) := \pi_{k-1} \circ \rho_k^B(\nu_k^\lambda) \circ \cdots \circ \rho_n^B(\nu_n^\lambda), \tag{6.2}$$

and denote the $U_\varepsilon(B_n)$-module associated with $(\phi_{k,n}(\lambda), V_{k,n})$ by $V_{k,n}(\lambda)$.

Let $m_n = (m_{1,n}, \cdots, m_{n,n}) \in \mathbb{Z}_l^n$, $\tilde{m}_{n-1} = (\tilde{m}_{1,n-1}, \cdots, \tilde{m}_{n-1,n-1}) \in \mathbb{Z}_l^{n-1}$, $w \in V_{k,n-1}$, $v = v_n(m_n) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes w \in V_{k,n}(\lambda)$. Then, by (4.2), (4.4), (4.9), (6.1), for any $1 < i < n$, we have

$$\begin{aligned}
e_{n,n}v &= [-m_{n,n}](v_n(m_n - \epsilon_{n,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes w), && (6.3)\\
e_{i,n}v &= [m_{i+1,n} - m_{i,n}](v_n(m_n - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes w)\\
&\quad +[\tilde{m}_{i-1,n-1} - \tilde{m}_{i,n-1}](v_n(m_n + \epsilon_{i+1,n} - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{i,n-1}) \otimes w)\\
&\quad +v_n(m_n + \epsilon_{i+1,n} - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{i,n-1} + \tilde{\epsilon}_{i-1,n-1}) \otimes (e_{i,n-1}w), && (6.4)\\
e_{1,n}v &= [2m_{2,n} - m_{1,n}]_{\varepsilon_1}(v_n(m_n - \epsilon_{1,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes w)\\
&\quad +[m_{1,n} - 2\tilde{m}_{1,n-1}]_{\varepsilon_1} v_n(m_n + \epsilon_{2,n} - \epsilon_{1,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{1,n-1}) \otimes w\\
&\quad +v_n(m_n + \epsilon_{2,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{1,n-1}) \otimes (e_{1,n-1}w). && (6.5)
\end{aligned}$$

Let $m = (m_{i,j})_{1 \leq i \leq j, k \leq j \leq n} \in \mathbb{Z}_l^{N_n - N_{k-1}}$, $\tilde{m} = (\tilde{m}_{i,j-1})_{1 \leq i \leq j-1, k \leq j \leq n} \in \mathbb{Z}_l^{N_{n-1} - N_{k-2}}$, where $N_i := \frac{1}{2}i(i+1)$ for $i \in \mathbb{N}$. We set

$$\begin{aligned}
v_{k,n}(m, \tilde{m}) &:= (\bigotimes_{j=k}^{n} v_j(m_{1,j}, \cdots m_{j,j})) \otimes (\bigotimes_{j=k}^{n} \tilde{v}_{j-1}(\tilde{m}_{1,j-1}, \cdots \tilde{m}_{j-1,j-1})),\\
v_{k,n}^{\mathbf{0}} &:= v_{k,n}(\mathbf{0}, \mathbf{0}). && (6.6)
\end{aligned}$$

Let $P(V_{k,n}(\lambda))$ as in Definition 3.3 (i).

Proposition 6.1. *For all $\lambda \in \mathbb{C}^{n-k+1}$, we obtain $P(V_{k,n}(\lambda)) = \mathbb{C}v_{k,n}^{\mathbf{0}}$.*

Proof. Since the actions of $e_{i,n}$ on $V_{k,n}(\lambda)$ do not depend on λ, we simply denote $V_{k,n}(\lambda)$ by $V_{k,n}$. By (6.4), (6.5), obviously, $\mathbb{C}v_{k,n}^{\mathbf{0}} \subset P(V_{k,n})$. So we shall prove $P(V_{k,n}) \subset \mathbb{C}v_{k,n}^{\mathbf{0}}$ by induction on n.

We assume $n = 1$. Then we have $k = 1$. Let $v = \sum_{m_{1,1} \in \mathbb{Z}_l} c(m_{1,1}) v(m_{1,1}) \in V_{1,1}$ ($c(m_{1,1}) \in \mathbb{C}$), and we assume $e_{1,1}v = 0$. Then, by (6.5), we get

$$0 = e_{1,1}v = \sum_{m_{1,1} \in \mathbb{Z}_l} c(m_{1,1})[-m_{1,1}]_{\varepsilon_1} v_{1,1}(m_{1,1} - 1).$$

Hence, we obtain $c(m_{1,1}) = 0$ if $m_{1,1} \neq 0$. Therefore we have $v = c(0)v_{1,1}(0) \in \mathbb{C}v_{1,1}(0) = \mathbb{C}v_{1,1}^{\mathbf{0}}$.

Now, we assume that $n > 1$ and we obtain the case of $(n-1)$. Let

$$v = \sum_{m_n \in \mathbb{Z}_l^n, \tilde{m}_{n-1} \in \mathbb{Z}_l^{n-1}} c(m_n, \tilde{m}_{n-1})(v_n(m_n) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes v_{m_n,\tilde{m}_{n-1}}) \in V_{k,n},$$

where $c(m_n, \tilde{m}_{n-1}) \in \mathbb{C}$, $v_{m_n,\tilde{m}_{n-1}} \in V_{k,n-1}$ ($V_{n,n-1} := \mathbb{C}v^{\mathbf{0}}_{n,n-1}$, $v^{\mathbf{0}}_{n,n-1} := 1$). We assume that $e_{i,n}v = 0$ for all $1 \le i \le n$.

First, we shall prove that $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_n \neq \mathbf{0}$. By (6.3), we get

$$0 = e_{n,n}v = \sum_{m_n,\tilde{m}_{n-1}} c(m_n, \tilde{m}_{n-1})[-m_{n,n}](v_n(m_n - \epsilon_{n,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes v_{m_n,\tilde{m}_{n-1}}).$$

Hence, we obtain $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{n,n} \neq 0$. Now, we assume that there exists a $2 \le i \le n-1$ such that $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{i+1,n} \neq 0, \cdots m_{n-1,n} \neq 0$ or $m_{n,n} \neq 0$. Then, by (6.4), we have

$$\begin{aligned}
0 = e_{i,n}v = \sum_{m_n,\tilde{m}_{n-1}} & c(m_n, \tilde{m}_{n-1})\{[-m_{i,n}](v_n(m_n - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes v_{m_n,\tilde{m}_{n-1}}) \\
& +[\tilde{m}_{i-1,n-1} - \tilde{m}_{i,n-1}](v_n(m_n + \epsilon_{i+1,n} - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{i,n-1}) \otimes v_{m_n,\tilde{m}_{n-1}}) \\
& +v_n(m_n + \epsilon_{i+1,n} - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{i,n-1} + \tilde{\epsilon}_{i-1,n-1}) \otimes (e_{i,n-1}v_{m_n,\tilde{m}_{n-1}})\}.
\end{aligned}$$

If $m_{i+1,n} = 0$, then the $(i+1,n)$-component of $(m_n - \epsilon_{i,n})$ is 0, and the one of $(m_n + \epsilon_{i+1,n} - \epsilon_{i,n})$ is 1. Thus, by the linearly independence, $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{i,n} \neq 0$. Therefore we obtain $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{2,n} \neq 0, \cdots m_{n-1,n} \neq 0$ or $m_{n,n} \neq 0$ inductively. Similarly, we have $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_{1,n} \neq 0$ by using $e_{1,n}v = 0$. Hence, we obtain $c(m_n, \tilde{m}_{n-1}) = 0$ if $m_n \neq \mathbf{0}$. Therefore we get

$$v = \sum_{\tilde{m}_{n-1} \in \mathbb{Z}_l^{n-1}} c(\mathbf{0}, \tilde{m}_{n-1})(v_n(\mathbf{0}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1}) \otimes v_{\mathbf{0},\tilde{m}_{n-1}}).$$

Moreover, we have

$$\begin{aligned}
0 &= e_{i,n}v \\
&= \sum_{\tilde{m}_{n-1}} c(\mathbf{0}, \tilde{m}_{n-1})\{[\tilde{m}_{i-1,n-1} - \tilde{m}_{i,n-1}](v_n(\epsilon_{i+1,n} - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{i,n-1}) \otimes v_{\mathbf{0},\tilde{m}_{n-1}}) \\
&\qquad +v_n(\epsilon_{i+1,n} - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{i,n-1} + \tilde{\epsilon}_{i-1,n-1}) \otimes (e_{i,n-1}v_{\mathbf{0},\tilde{m}_{n-1}})\} \ (i \neq 1), \\
0 &= e_{1,n}v \\
&= \sum_{\tilde{m}_{n-1}} c(\mathbf{0}, \tilde{m}_{n-1})\{[-2\tilde{m}_{1,n-1}]_{\varepsilon_1}(v_n(\epsilon_{2,n} - \epsilon_{1,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{1,n-1}) \otimes v_{\mathbf{0},\tilde{m}_{n-1}}) \\
&\qquad +v_n(\epsilon_{2,n}) \otimes \tilde{v}_{n-1}(\tilde{m}_{n-1} - \tilde{\epsilon}_{1,n-1}) \otimes (e_{1,n-1}v_{\mathbf{0},\tilde{m}_{n-1}})\}.
\end{aligned}$$

Then, by the similar manner to the above proof, we obtain $c(\mathbf{0}, \tilde{m}_{n-1}) = 0$ if $\tilde{m}_{n-1} \neq \mathbf{0}$ by using $e_{i,n}v = 0$ $(1 \le i \le n)$. Finally, we get

$$\begin{aligned}
& v = c(\mathbf{0},\mathbf{0})(v_n(\mathbf{0}) \otimes \tilde{v}_{n-1}(\mathbf{0}) \otimes v_{\mathbf{0},\mathbf{0}}), \\
& 0 = e_{i,n}v = c(\mathbf{0},\mathbf{0})v_n(\epsilon_{i+1,n} - \epsilon_{i,n}) \otimes \tilde{v}_{n-1}(-\tilde{\epsilon}_{i,n-1} + \tilde{\epsilon}_{i-1,n-1}) \otimes (e_{i,n-1}v_{\mathbf{0},\mathbf{0}}) \ (i \neq 1, n), \\
& 0 = e_{1,n}v = c(\mathbf{0},\mathbf{0})v_n(\epsilon_{2,n}) \otimes \tilde{v}_{n-1}(-\tilde{\epsilon}_{1,n-1}) \otimes (e_{1,n-1}v_{\mathbf{0},\mathbf{0}}).
\end{aligned}$$

Hence $e_{i,n-1}v_{\mathbf{0},\mathbf{0}} = 0$ in $V_{k,n-1}$ for all $1 \le i \le n-1$ if $c(\mathbf{0},\mathbf{0}) \neq 0$. So, by the assumption of the induction on n, we obtain $v_{\mathbf{0},\mathbf{0}} \in \mathbb{C}v^{\mathbf{0}}_{k,n-1}$ if $c(\mathbf{0},\mathbf{0}) \neq 0$. Therefore

$$v \in \mathbb{C}(v_n(\mathbf{0}) \otimes \tilde{v}_{n-1}(\mathbf{0}) \otimes v^{\mathbf{0}}_{k,n-1}) = \mathbb{C}v^{\mathbf{0}}_{k.n}.$$

$\square$

Let $m = (m_{i,j})_{1\le i\le j, k\le j\le n} \in \mathbb{Z}_l^{N_n - N_{k-1}}$, $\tilde{m} = (\tilde{m}_{i,j-1})_{1\le i\le j-1, k\le j\le n} \in \mathbb{Z}_l^{N_{n-1}-N_{k-2}}$. For $1 \le i \le n$, $\max(k,i) \le j \le n$, we define

$$\begin{aligned}
&\nu_{i,j}(m,\tilde{m}) := m_{i+1,j} - 2m_{i,j} + m_{i-1,j} + \tilde{m}_{i-2,j-1} - 2\tilde{m}_{i-1,j-1} + \tilde{m}_{i,j-1} \quad (i \neq 1),\\
&\nu_{1,j}(m,\tilde{m}) := m_{2,j} - m_{1,j} + \tilde{m}_{1,j-1},\\
&\mu_{i,i-1}(m,\tilde{m}) := \xi(i>k)(m_{i-1,i-1} + \tilde{m}_{i-2,i-2}),\\
&\mu_{i,j}(m,\tilde{m}) := \mu_{i,i-1}(m,\tilde{m}) + \sum_{r=\max(k,i)}^{j} \nu_{i,r}(m,\tilde{m}),
\end{aligned} \tag{6.7}$$

where

$$\xi(i>j) := \begin{cases} 1 & (i>j) \\ 0 & (i\le j), \end{cases} \qquad \xi(i\ge j) := \begin{cases} 1 & (i\ge j) \\ 0 & (i<j). \end{cases} \tag{6.8}$$

Let $v_{k,n}(m,\tilde{m})$ be as in (6.6). Then, by (4.2), (4.10), (4.25), (6.1), we obtain

$$t_{i,n} v_{k,n}(m,\tilde{m}) \quad = \quad \varepsilon_i^{\mu_{i,n}(m,\tilde{m}) + \xi(i\ge k)\lambda_i} v_{k,n}(m,\tilde{m}). \tag{6.9}$$

In particular, we obtain the following lemma.

Lemma 6.2. *For any $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{C}^{n-k+1}$, $i \in I$, we obtain*

$$t_{i,n} v_{k,n}^{\boldsymbol{0}} = \varepsilon_i^{\xi(i\ge k)\lambda_i} v_{k,n}^{\boldsymbol{0}} \quad \text{in } V_{k,n}(\lambda).$$

Let $m = (m_{i,j})_{1\le i\le j, k\le j\le n} \in \mathbb{Z}_l^{N_n - N_{k-1}}$, $\tilde{m} = (\tilde{m}_{i,j-1})_{1\le i\le j-1, k\le j\le n} \in \mathbb{Z}_l^{N_{n-1}-N_{k-2}}$, $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{C}^{n-k+1}$. For $i \in I$, $\max(k,i) \le j \le n$, we define $y_{i,j}$, $\tilde{y}_{i-1,j-1} \in \mathrm{End}(V_{k,n}(\lambda))$ by: for $v = v_{k,n}(m,\tilde{m})$,

$$\begin{aligned}
\tilde{y}_{i-1,j-1} v &:= [\tilde{m}_{i-1,j-1} - \tilde{m}_{i,j-1} - \mu_{i,j-1}(m,\tilde{m}) - \xi(i\ge k)\lambda_i] v_{k,n}(m, \tilde{m} + \tilde{\epsilon}_{i-1,j-1}),\\
y_{i,j} v &:= [m_{i+1,j} - m_{i,j} - \mu_{i,j}(m,\tilde{m}) - \xi(i\ge k)\lambda_i] v_{k,n}(m + \epsilon_{i,j}, \tilde{m}) \quad (i \neq 1),\\
y_{1,j} v &:= [m_{1,j} - 2\tilde{m}_{1,j-1} - \mu_{1,j}(m,\tilde{m}) - \xi(1\ge k)\lambda_i]_{\varepsilon_1} v_{k,n}(m + \epsilon_{1,j}, \tilde{m}),
\end{aligned} \tag{6.10}$$

where $\tilde{y}_{0,j-1} := 0$. We set

$$Y_{k,n} := \{ y_{i,j}, \tilde{y}_{i-1,j-1} \,|\, i \in I, \max(k,i) \le j \le n \}.$$

Then, by (4.2), (4.4), (4.11), (6.1), we have

$$f_{i,n} v_{k,n}(m,\tilde{m}) = \sum_{j=\max(k,i)}^{n} (y_{i,j} + \tilde{y}_{i-1,j-1}) v_{k,n}(m,\tilde{m}) \quad \text{in } V_{k,n}(\lambda). \tag{6.11}$$

Let $p_{\mathbf{0}} : V_{k,n}(\lambda) \longrightarrow \mathbb{C} v_{k,n}^{\mathbf{0}}$ be the projection.

Lemma 6.3. *Let $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{Z}^{n-k+1}$.*

(a) For all $r \in \mathbb{N}$, $g_1, \cdots, g_r \in Y_{k,n}$, we have

$$p_{\boldsymbol{0}}(g_1 \cdots g_r v_{k,n}^{\boldsymbol{0}}) = 0 \quad in\, V_{k,n}(\lambda).$$

(b) For all $r \in \mathbb{N}$, $i_1, \cdots, i_r \in I$, we have

$$p_{\boldsymbol{0}}(f_{i_1,n} \cdots f_{i_r,n} v_{k,n}^{\boldsymbol{0}}) = 0 \quad in\, V_{k,n}(\lambda).$$

Proof. If we can prove (a), then we obtain (b) by (6.11). So we shall prove (a).

Let $r \in \mathbb{N}$, $g_1, \cdots, g_r \in (Y_{k,n} - \{0\})$ and set $g := g_1 \cdots g_r$. For $y \in Y_{k,n}$, we set

$$s(y) := \#\{1 \leq r' \leq r \,|\, g_{r'} = y\} \geq 0, \quad s_j := \sum_{i=1}^{j} (s(y_{i,j}) + s(\tilde{y}_{i-1,j-1})) \quad (k \leq j \leq n),$$

$$m_g := \sum_{j=k}^{n} \sum_{i=1}^{j} (s(y_{i,j})\epsilon_{i,j} + s(\tilde{y}_{i-1,j-1})\tilde{\epsilon}_{i-1,j-1}), \quad W_g := \bigoplus_{r'=1}^{r} \mathbb{C}(g_{r'} g_{r'+1} \cdots g_r v_{k,n}^{\mathbf{0}}).$$

Then, $gv_{k,n}^{\mathbf{0}} \in \mathbb{C}v_{k,n}(m_g)$ by (6.10). Since $\sum_{j=k}^{n} s_j = r > 0$, there exists a $k \leq j \leq n$ such that $s_k = \cdots = s_{j-1} = 0$ and $s_j > 0$. Then, $s(y_{p,q}) = s(\tilde{y}_{p-1,q-1}) = 0$ for all $k \leq q < j$, $1 \leq p \leq q$. Thus, for any $1 \leq r' \leq r$, there exist $m^{(r')} \in \mathbb{Z}_l^{N_n - N_{k-1}}$, $\tilde{m}^{(r')} \in \mathbb{Z}_l^{N_{n-1} - N_{k-2}}$ such that $g_{r'} g_{r'+1} \cdots g_r v_{k,n}^{\mathbf{0}} \in \mathbb{C}v_{k,n}(m^{(r')}, \tilde{m}^{(r')})$ and $m_{p,q}^{(r')} = \tilde{m}_{p-1,q-1}^{(r')} = 0$ for all $k \leq q < j$, $1 \leq p \leq q$. Hence, by (6.7), (6.10), in W_g,

$$\begin{aligned}
y_{i,j} v_{k,n}(m, \tilde{m}) &= [m_{i+1,j} - m_{i,j} - \nu_{i,j}(m, \tilde{m}) - \xi(i \geq k)\lambda_i] v_{k,n}(m + \epsilon_{i,j}, \tilde{m}), \\
y_{1,j} v_{k,n}(m, \tilde{m}) &= [m_{1,j} - 2\tilde{m}_{1,j-1} - \xi(1 \geq k)\lambda_1]_{\varepsilon_1} v_{k,n}(m + \epsilon_{1,j}, \tilde{m}), \\
\tilde{y}_{i-1,j-1} v_{k,n}(m, \tilde{m}) &= [\tilde{m}_{i-1,j-1} - \tilde{m}_{i,j-1} - \xi(i \geq k)\lambda_i] v_{k,n}(m, \tilde{m} + \tilde{\epsilon}_{i-1,j-1}), \quad (6.12)
\end{aligned}$$

for $2 \leq i \leq j$, where $\xi(i \geq j)$ as in (6.8).

On the other hand, since $s_j > 0$, there exist i $(1 \leq i \leq j)$ such that $s(y_{i,j}) > 0$ or $s(\tilde{y}_{i-1,j-1}) > 0$. Now, we assume $s(\tilde{y}_{j-1,j-1}) > 0$. Let r_1 $(1 \leq r_1 \leq r)$ such that $g_{r_1} = \tilde{y}_{j-1,j-1}$ and $g_{r_1+1}, \cdots, g_r \neq \tilde{y}_{j-1,j-1}$. Then, by (6.10), $\tilde{m}_{j-1,j-1}^{(r_1+1)} = 0$. Hence, by (6.12),

$$g_{r_1} g_{r_1+1} \cdots g_r v_{k,n}^{\mathbf{0}} \in \mathbb{C}[-\lambda_j] v_{k,n}(m^{(r_1+1)}, \tilde{m}^{(r_1+1)} + \tilde{\epsilon}_{j-1,j-1}).$$

Thus, by the similar way to the proof of Case 1 in Lemma 5.2, we have

$$gv_{k,n}^{\mathbf{0}} \in \mathbb{C}[-\lambda_j + s(\tilde{y}_{j-1,j-1}) - 1] \cdots [-\lambda_j + 1][-\lambda_j] v_{k,n}(m_g),$$

and $p_{\mathbf{0}}(gv_{k,n}^{\mathbf{0}}) = 0$. Similarly, if there exists i $(2 \leq i \leq j-1)$ such that $s(\tilde{y}_{j-1,j-1}) = \cdots = s(\tilde{y}_{i,j-1}) = 0, s(\tilde{y}_{i-1,j-1}) > 0$, then we have

$$\tilde{y}_{i-1,j-1} v_{k,n}(m, \tilde{m}) = [\tilde{m}_{i-1,j-1} - \delta_{i \geq k}\lambda_i] v_{k,n}(m, \tilde{m} + \tilde{\epsilon}_{i-1,j-1}) \quad \text{in } W_g,$$

and $p_{\mathbf{0}}(gv_{k,n}^{\mathbf{0}}) = 0$. If $s(\tilde{y}_{1,j-1}) = \cdots = s(\tilde{y}_{j-1,j-1}) = 0$, and there exists i $(1 \leq i \leq j)$ such that $s(y_{1,j}) = \cdots = s(y_{i-1,j}) = 0, s(y_{i,j}) > 0$, then we obtain

$$y_{i,j} v_{k,n}(m, \tilde{m}) = [m_{i,j} - \delta_{i \geq k}\lambda_i]_{\varepsilon_i} v_{k,n}(m + \epsilon_{i,j}, \tilde{m}) \quad \text{in } W_g,$$

and $p_{\mathbf{0}}(gv_{k,n}^{\mathbf{0}}) = 0$.

Consequently, we obtain that $p_{\mathbf{0}}(gv_{k,n}^{\mathbf{0}}) = 0$ if $s_j > 0$. It amount to $p_{\mathbf{0}}(gv_{k,n}^{\mathbf{0}}) = 0$. □

Lemma 6.4. *For all $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{Z}^{n-k+1}$, $\alpha \in \Delta_+$, we have*

$$f_{\alpha,n}^{l} v_{k,n}^{\mathbf{0}} = 0 \quad \text{in } V_{k,n}(\lambda).$$

Proof. By Lemma 6.3(b) and Proposition 2.4, we obtain $p_{\mathbf{0}}(f_{\alpha,n}^{l} v_{k,n}^{\mathbf{0}}) = 0$. On the other hand, by Proposition 2.3, 6.1,

$$e_{i,n} f_{\alpha,n}^{l} v_{k,n}^{\mathbf{0}} = f_{\alpha,n}^{l} e_{i,n} v_{k,n}^{\mathbf{0}} = 0,$$

for all $i \in I$. Hence, by Proposition 6.1, we get $f_{\alpha,n}^{l} v_{k,n}^{\mathbf{0}} \in \mathbb{C} v_{k,n}^{\mathbf{0}}$. Therefore

$$f_{\alpha,n}^{l} v_{k,n}^{\mathbf{0}} = p_{\mathbf{0}}(f_{\alpha,n}^{l} v_{k,n}^{\mathbf{0}}) = 0.$$

□

Now, we construct nilpotent $U_\varepsilon(B_n)$-modules (see §3). For $\lambda \in \mathbb{C}^{n-k+1}$, let $L_{k,n}(\lambda)$ be the $U_\varepsilon(B_n)$-submodule of $V_{k,n}(\lambda)$ generated by $v_{k,n}^{\mathbf{0}}$.

Theorem 6.5. *For any $k \in I$, $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{Z}_l^{n-k+1}$, $L_{k,n}(\lambda)$ is isomorphic to $L_\varepsilon^{nil}(0, \cdots, 0, \lambda_k, \cdots, \lambda_n)$ as $U_\varepsilon(B_n)$-module.*

Proof. By Proposition 6.1 and Lemma 6.2, $L_{k,n}(\lambda)$ is a highest weight $U_\varepsilon(B_n)$-module with highest weight $(0, \cdots, 0, \lambda_k, \cdots \lambda_n)$. On the other hand, by Lemma 6.4, $f_{\alpha,n}^{l} v_{k,n}^{\mathbf{0}} = 0$ for all $\alpha \in \Delta_+$. Moreover, by Proposition 2.4, 6.1, $e_{\alpha,n}^{l} v_{k,n}^{\mathbf{0}} = 0$ for all $\alpha \in \Delta_+$. Hence, by Proposition 2.3, $e_{\alpha,n}^{l} = f_{\alpha,n}^{l} = 0$ on $L_{k,n}(\lambda)$ for all $\alpha \in \Delta_+$. Thus $L_{k,n}(\lambda)$ is a nilpotent $U_\varepsilon(B_n)$-module. Therefore, by Proposition 6.1 and Proposition 3.5, we obtain this theorem. □

In particular, if $k = 1$ then we obtain all finite dimensional irreducible nilpotent $U_\varepsilon(B_n)$-modules of type 1.

7 Other cases

In this section, we construct $U_\varepsilon(\mathfrak{g}_n)$-modules $L_\varepsilon^{\mathrm{nil}}(\lambda)$ in the case of $\mathfrak{g}_n = A_n$, C_n or D_n inductively by using the Schnizer-homomorphisms in Theorem 4.1(a), (c), (d).

We set $a_n^{(0)} = (a_{i,n}^{(0)})_{i=1}^n$, $\tilde{a}_n^{(0)} = (\tilde{a}_{i,n}^{(0)})_{i=1}^n$, $b_n^{(0)} = (b_{i,n}^{(0)})_{i=1}^n$, $\tilde{b}_n^{(0)} = (\tilde{b}_{i,n}^{(0)})_{i=1}^n \in \mathbb{C}^n$ by,

$$\begin{aligned}
&a_{i,n}^{(0)} := \tilde{a}_{i,n}^{(0)} := 1 \quad (\mathfrak{g}_n = A_n,\, C_n,\, D_n),\\
&b_{i,n}^{(0)} := i \quad (\mathfrak{g}_n = A_n), \quad b_{i,n}^{(0)} := n - i + 1 \quad (\mathfrak{g}_n = C_n),\\
&b_{i,n}^{(0)} := n - i + 1 \quad (i \neq 1), \quad b_{1,n}^{(0)} := n - 1 \quad (n \neq 1), \quad b_{1,1}^{(0)} := 1 \quad (\mathfrak{g}_n = D_n),\\
&\tilde{b}_{i,n}^{(0)} := n + i - 1 \quad (\mathfrak{g}_n = C_n,\, D_n).
\end{aligned}$$

We fix $k \in I$. For $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{C}^{n-k+1}$, we define $\nu^\lambda = (\nu_k^\lambda, \cdots, \nu_n^\lambda) \in \mathbb{C}^{n-k+1}$ by

$$\begin{aligned}
&\nu_i^\lambda := -i - 1 - \frac{1}{i}\sum_{j=k}^{i}(j\lambda_j) \quad (\mathfrak{g}_n = A_n), \quad \nu_i^\lambda := -2i - \sum_{j=k}^{i}\lambda_j \quad (\mathfrak{g}_n = C_n),\\
&\nu_i^\lambda := -2i + 2 - \sum_{j=k}^{i}\lambda_j \quad (k \geq 3), \quad \nu_i^\lambda := -2i + 3 - \frac{1}{2}\lambda_2 - \sum_{j=3}^{i}\lambda_j \quad (k = 2),\\
&\nu_i^\lambda := -2i + 1 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 - \sum_{j=3}^{i}\lambda_j \quad (k = 1), \quad (\mathfrak{g}_n = D_n),
\end{aligned}$$

where $k \leq i \leq n$. For $\lambda \in \mathbb{C}$, we set $\rho_n^A(\lambda) := \rho_n^A(a_n^{(0)}, b_n^{(0)}\lambda)$, $\rho_n^C(\lambda) := \rho_n^C(a_n^{(0)}, \tilde{a}_{n-1}^{(0)}, b_n^{(0)}, \tilde{b}_{n-1}^{(0)}, \lambda)$, and $\rho_n^D(\lambda) := \rho_n^D(a_n^{(0)}, \tilde{a}_{n-2}^{(0)}, b_n^{(0)}, \tilde{b}_{n-2}^{(0)}, \lambda)$ (see Theorem 4.1(a), (c), (d)). We define

$$\begin{aligned}
&V_{k,n} := \bigotimes_{j=k}^{n} V_j \quad (\mathfrak{g}_n = A_n), \quad V_{k,n} := \bigotimes_{j=k}^{n}(V_j \otimes \widetilde{V}_{j-1}) \quad (\mathfrak{g}_n = C_n),\\
&V_{k,n} := \bigotimes_{j=k}^{n}(V_j \otimes \widetilde{V}_{j-2}) \quad (\mathfrak{g}_n = D_n).
\end{aligned}$$

For $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{C}^{n-k+1}$, we define $U_\varepsilon(\mathfrak{g}_n)$-representations $\phi_{k,n} := \phi_{k,n}(\lambda) : U_\varepsilon(\mathfrak{g}_n) \longrightarrow \mathrm{End}(V_{k,n})$ by

$$\phi_{k,n}(\lambda) := \pi_{k-1} \circ \rho_k^{\mathfrak{g}}(\nu_k^\lambda) \circ \cdots \circ \rho_n^{\mathfrak{g}}(\nu_n^\lambda).$$

We denote the $U_\varepsilon(\mathfrak{g}_n)$-module associated with $(\phi_{k,n}(\lambda), V_{k,n})$ by $V_{k,n}(\lambda)$. We set

$$v_{k,n}^{\mathbf{0}} := \bigotimes_{j=k}^{n} v_j(0, \cdots, 0) \quad (\mathfrak{g}_n = A_n),$$

$$v_{k,n}^{\mathbf{0}} := \bigotimes_{j=k}^{n} (v_j(0, \cdots, 0) \otimes \tilde{v}_{j-1}(0, \cdots, 0)) \quad (\mathfrak{g}_n = C_n),$$

$$v_{k,n}^{\mathbf{0}} := \bigotimes_{j=k}^{n} (v_j(0, \cdots, 0) \otimes \tilde{v}_{j-2}(0, \cdots, 0)) \quad (\mathfrak{g}_n = D_n).$$

For $\lambda \in \mathbb{C}^{n-k+1}$, let $L_{k,n}(\lambda)$ be the $U_\varepsilon(\mathfrak{g}_n)$-submodule of $V_{k,n}(\lambda)$ generated by $v_{k,n}^{\mathbf{0}}$. Then we have the following theorem by the similar way to §6.

Theorem 7.1. *Let $\mathfrak{g}_n = A_n$, C_n or D_n. Then, for any $k \in I$, $\lambda = (\lambda_k, \cdots, \lambda_n) \in \mathbb{Z}_l^{n-k+1}$, $L_{k,n}(\lambda)$ is isomorphic to $L_l^{nil}(0, \cdots, 0, \lambda_k, \cdots \lambda_n)$ as $U_\varepsilon(\mathfrak{g}_n)$-module. In particular, for any $\lambda \in \mathbb{C}^{n-k+1}$, we have $P(V_{k,n}(\lambda)) = \mathbb{C}v_{k,n}^{\mathbf{0}}$.*

Consequently, we obtain all finite dimensional irreducible nilpotent modules of type 1 inductively by using the Schnizer homomorphisms for quantum algebras at roots of unity of type A_n, B_n, C_n, D_n or G_2.

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